ME8692 – FINITE ELEMENT ANALYSIS

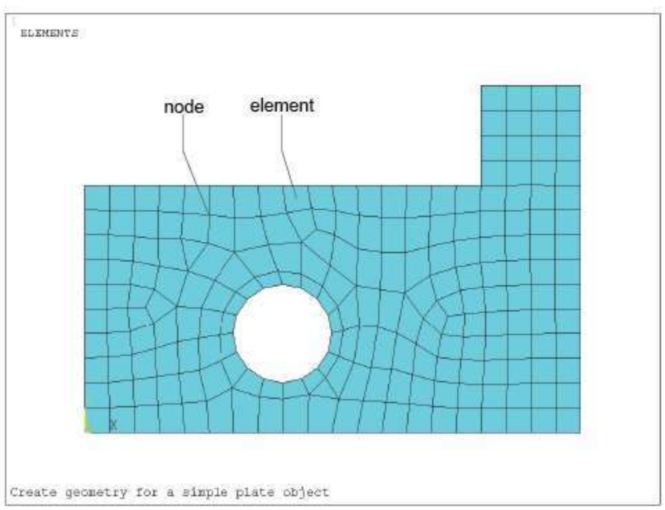
UNIT NOTES

UNIT-I INTRODUCTION

Basic principles

The basic principles underlying the FEM are relatively simple. Consider a body or engineering component through which the distribution of a field variable, e.g. displacement or stress, is required. Examples could be a component under load, temperatures subject to a heat input, etc. The body, i.e. a one-, two- or three-dimensional solid, is modelled as being hypothetically subdivided into an assembly of small parts called *elements* – 'finite elements'. The word 'finite' is used to describe the limited, or finite, number of degrees of freedom used to model the behaviour of each element. The elements are assumed to be connected to one another, but only at interconnected joints, known as *nodes* . It is important to note that the elements are notionally small regions, not separate entities like bricks, and there are no cracks or surfaces between them.

The complete set, or assemblage of elements, is known as a *mesh* . The process of representing a component as an assemblage of finite elements, known as discretisation, is the first of many key steps in understanding the FEM of analysis. An example is illustrated in Figure 1. This is a plate-type component modelled with a number of mostly rectangular(ish) elements with a uniform thickness (into the page or screen) that could be, say, 2 mm.



Weighted Residual Methods Problems :

1. $\frac{d^2y}{dx^2} + 50 = 0.$ $\quad 0 \leq x \leq 10.$

trail soln $y = a_1 x(10 - x).$

Boundary conditions $y(0) = 0$

$y(10) = 0.$

Find the value of the parameter a_1 by following methoc

(i) Point collocation method (ii) Sub domain (iv) Least square

(iii) Galerkin.

SOL : Verify wheather trail fn satisfies the boundary conditions or not.

$y = a_1 x(10 - x)$

$$\left.\begin{array}{ll} x = 0 & y = 0 \\ x = 10 & y = 0 \end{array}\right\} \longrightarrow \text{Satisfies Boundary condition}$$

(i) Point collocation. Method :

$y = a_1 x(10 - x).$

$= a_1 (10x - x^2).$

$\frac{dy}{dx} = a_1(10 - 2x).$

$\frac{d^2y}{dx^2} = -2a_1 \longrightarrow (1).$

$R = \frac{d^2y}{dx^2} + 50 = -2a_1 + 50 = 0. \longrightarrow (2).$

$2a_1 = 50.$

$a_1 = 25.$

$y = 25x(10 - x). \longrightarrow (3).$

(ii) Sub domain method :

$$\int_0^{10} R\,dx = 0.$$

$$\int_0^{10} (-2a_1 + 50)\,dx = 0.$$

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$$\int_0^{10} -2a_1\,dx + 50\,dx = 0.$$

$$\left[-2a_1 x + 50x\right]_0^{10} = 0.$$

$$-2a_1(10) + 50(10) = 0.$$

$$20a_1 = 500.$$

$$a_1 = 25.$$

$$y = 25x(10-x).$$

(iii). Least Square Method :

$$I = \int_0^{10} R^2\,dx.$$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} R \cdot \frac{\partial R}{\partial a_1}\,dx.$$

$$R = -2a_1 + 50.$$

$$\frac{\partial R}{\partial a_1} = -2. \longrightarrow (4).$$

$$\frac{\partial I}{\partial a_1} = 0.$$

$$\int_0^{10} (-2a_1 + 50)(-2)\,dx = 0.$$

$$\int_0^{10} -2a_1\,dx + 50\,dx = 0.$$

$$\left[-2a_1 x + 50x\right]_0^{10} = 0.$$

$$-20a_1 + 500 = 0.$$

$$a_1 = 25.$$

$$y = 25x(10-x). \longrightarrow (5).$$

(iv). Galerkin method :

$$\int_0^{10} w_i\, R \cdot dx = 0.$$

$$y = w_i = a_1 x(10-x).$$

$$a_1 \int_0^{10} x(10-x)(-2a_1+50)\,dx = 0.$$

$$a_1 \cdot \int_0^{10} (10x - x^2)(-2a_1+50)\,dx = 0.$$

$$a_1 \int_0^{10} (-20a_1 x + 500x + 2a_1 x^2 - 50x^2)\,dx = 0.$$

$$a_1 \left[-20a_1 \cdot \frac{x^2}{2} + 500 \cdot \frac{x^2}{2} + 2a_1 \frac{x^3}{3} - 50 \cdot \frac{x^3}{3}\right]_0^{10} = 0.$$

$$a_1 \left[-\frac{20a_1}{2}(10^2) + \frac{500}{2}(10^2) + \frac{2a_1}{2}(10^3) - \frac{50}{3}(10^3)\right] = 0.$$

$$-1000a_1 - 667.67a_1 + 2500 - 16666.67 = 0.$$

$\Rightarrow -1000\, a_1 - 666.67\, a_1 + 8333.33 = 0.$

$$-333.33\, a_1 = -8333.33.$$

$$a_1 = 25.$$

$$y = 25\, x(10 - x).$$

$\therefore$ We know that the value of parameter a_1 is same for all 4 methods.

$$\boxed{a_1 = 25}.$$

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2. The following eq is available for the

$$\frac{d^2y}{dx^2} - 10x^2 = 5. \quad 0 \le x \le 1.$$

B.C : $y(0) = 0$

$y(1) = 0.$

By using Galerkin method of weighted residuals to find an approx sol. of the above diff eqn and also compare with exact sol.

Sol :- Galerkin method (Approx sol.).

$$y = a_1 x(x-1).$$

$$y = a_1 (x^2 - x).$$

$$\frac{dy}{dx} = a_1 (2x-1) \qquad \frac{d^2y}{dx^2} = a_1(2) \longrightarrow (1).$$

$$R = \frac{d^2y}{dx^2} - 10x^2 - 5.$$

$$R = 2a_1 - 10x^2 - 5 \longrightarrow (2).$$

$$\int_0^1 w_i . R\, dx = 0.$$

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$$y = w_i = a_1 x(x-1).$$

$$a_1 \int_0^1 (x^2 - x).(2a_1 - 10x^2 - 5)\, dx = 0.$$

$$a_1 \int_0^1 (2a_1 x^2 - 10x^4 - 5x^2 - 2a_1 x + 10x^3 + 5x)\, dx = 0.$$

$$2a_1 \left[\frac{x^3}{3}\right]_0^1 - 10\left[\frac{x^5}{5}\right]_0^1 - 5\left[\frac{x^3}{3}\right]_0^1 - 2a_1\left[\frac{x^2}{2}\right]_0^1 + 10\left[\frac{x^4}{4}\right]_0^1 + 5\left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\Rightarrow \frac{2a_1}{3} - \frac{10}{5} - \frac{5}{3} - \frac{2a_1}{2} + \frac{10}{4} + \frac{5}{2} = 0.$$

$$\Rightarrow 0.666 a_1 - 2 - 1.666 - a_1 + 2.5 + 2.5 = 0.$$

$$-0.33 a_1 + 1.334 = 0.$$

$$a_1 = 4.03. \simeq a_1 = 3.99 \simeq \boxed{a_1 = 4.}$$

$$\boxed{y = 4x(x-1)} \longrightarrow ③.$$

$\Downarrow$

Approx Sol:

(ii) Exact Sol :-

$$\frac{d^2y}{dx^2} = 10x^2 + 5. \quad \text{(Given)}.$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2} = \frac{10x^3}{3} + 5x + c.$$

$$y = \int \frac{dy}{dx} = \frac{10x^4}{3 \times 4} + \frac{5x^2}{2} + c_1 x + c_2.$$

$$\boxed{y = 0.833\, x^4 + 2.5x^2 + c_1 x + c_2} \longrightarrow ④.$$

Apply B.C $x = 0$ $y = 0$.

$$\boxed{c_2 = 0}.$$

$x = 1$. $y = 0$.

$$0.833 + 2.5 + c_1 + 0 = 0. \quad \boxed{c_1 = -3.333}.$$

Sub c_1 + c_2 in eqn (4).

$$y = 0.833x^4 + 2.5x^2 - 3.333x \longrightarrow (5).$$

$\Downarrow$

Exact sol.

RESULT :

(1). Approx. Sol $\rightarrow$ $y = 4x(x-1)$

(2). Exact sol $\rightarrow$ $y = 0.833x^4 + 2.5x^2 - 3.333x$.

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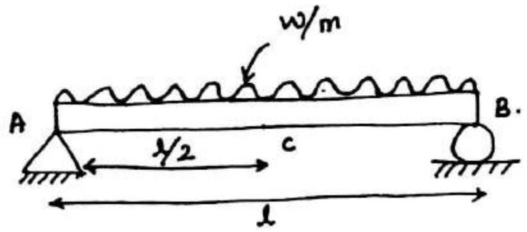
3. Find the deflection at the centre of a simply supported Beam of span length 'l' subjected to uniformly distributed Load throughout its length, Find using (i) point collocation (ii) Sub domain (iii) Least squares method (iv) Galerkin's method.



Sol :

Governing eq. :- $\boxed{EI.\dfrac{d^4y}{dx^4} - w = 0.}$ $0 \le x \le l$.

$x = 0$ $x = l$.

$E \rightarrow$ Young's Modulus.

$I \rightarrow$ Moment of Inertia.

<u>Trail sol</u> :- $\boxed{y = a\sin\dfrac{\pi x}{l}}$

it satisfies the boundary condition.

$$\frac{dy}{dx} = a.\frac{\pi}{l}.\cos\frac{\pi x}{l} \qquad \frac{d^3y}{dx^3} = -a.\frac{\pi^3}{l^3}\cos\frac{\pi x}{l}.$$

$$\frac{d^2y}{dx^2} = -a.\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}. \qquad \frac{d^4y}{dx^4} = a.\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}.$$

Sub $\frac{d^4y}{dx^4}$ value in Governing eqn.

$$EI.\left(a.\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}\right)-w=0.$$

$$\boxed{R = EI.\left(a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}\right)-w} \longrightarrow (1).$$

(i) <u>Point Collocation method</u> :-

$$R=0.$$

$$EI.\left(a\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}\right)-w=0$$

$$EI.\left(a.\frac{\pi^4}{l^4}\sin\frac{\pi x}{l}\right)=w.$$

To get max deflection $\boxed{x = l/2}$ → (Centre of beam).

$$EI.a.\frac{\pi^4}{l^4}\sin\frac{\pi}{l}.\frac{l}{2}=w.$$

$$EI.a.\frac{\pi^4}{l^4}(1)=w.$$

$$\boxed{a=\frac{wl^4}{\pi^4.EI}} \longrightarrow (2).$$

Sub the value 'a' in trial sol.

$$y=\frac{wl^4}{\pi^4.EI}.\sin\frac{\pi x}{l}.$$

$$y_{max}=\frac{wl^4}{\pi^4 EI}\sin\frac{\pi}{l}.\frac{l}{2}. \quad \Rightarrow x = l/2$$

$$y_{max}=\frac{wl^4}{\pi^4 EI}.$$

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$$y_{max} = \frac{0.010 \cdot w l^4}{EI}$$

$$\boxed{y_{max} = \frac{w l^4}{97.4\ EI}} \longrightarrow (3).$$

(ii) <u>Sub domain collocation method.</u>

$$\int_0^l R \cdot dx = 0$$

$$\int_0^l \left(EI \cdot a \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right) dx = 0.$$

$$\left[a\,EI \cdot \frac{\pi^4}{l^4} \left(\frac{-\cos \frac{\pi x}{l}}{\pi / l} \right) - w x \right]_0^l = 0.$$

$$\left[a\,EI \cdot \frac{\pi^3}{l^3} \left(-\cos \frac{\pi x}{l} \right) - w x \right]_0^l = 0.$$

$$\Rightarrow \quad -a\,EI \frac{\pi^3}{l^3} (\cos \pi - \cos 0) - w l = 0.$$

$$-a\,EI \frac{\pi^3}{l^3} (-1 - 1) - w l = 0.$$

$$2a\,EI \cdot \frac{\pi^3}{l^3} = w l \implies a = \frac{w l^4}{2\pi^3 EI}$$

$$\boxed{a = \frac{w l^4}{62\ EI}} \longrightarrow (7).$$

Sub the value a in trial sol

$$y = \frac{w l^4}{62 EI} \sin \frac{\pi x}{l}.$$

$$y_{max} = \frac{w l^4}{62 EI} \sin \frac{\pi}{l} \cdot \frac{l}{2}. \quad \boxed{x = l/2.}$$

$$y_{max} = \frac{w l^4}{62 EI}. \longrightarrow (4). \quad \boxed{y_{max} = \frac{0.016\, w l^4}{EI.}}$$

(ii) <u>Least Squares Method</u>:-

$$I = \int_0^l R^2 dx.$$

$$I = \int_0^l \left(a EI \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} - w \right)^2 dx.$$

$$= \int_0^l \left[a^2 E^2 I^2 \frac{\pi^8}{l^8} \sin^2 \frac{\pi x}{l} + w^2 - 2 a EI w \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} \right] dx.$$

$$\boxed{\sin^2 \frac{\pi x}{l} = \frac{1 - \cos\left(\frac{2\pi x}{l}\right)}{2}}$$

$$= \int_0^l \left[a^2 E^2 I^2 \frac{\pi^8}{l^8} \left(\frac{1 - \cos\left(\frac{2\pi x}{l}\right)}{2} \right) + w^2 - 2 a EI w \frac{\pi^4}{l^4} \sin \frac{\pi x}{l} \right] dx.$$

$$= \left[a^2 E^2 I^2 \frac{\pi^8}{l^8} \left(\frac{1}{2} x - \frac{1}{2} \sin\left(\frac{2\pi x}{l}\right) \left(\frac{l}{2\pi}\right) \right) + w^2 x - 2 a EI w \frac{\pi^4}{l^4} x - \cos\left(\frac{\pi x}{l}\right) \left(\frac{l}{\pi}\right) \right]_0^l$$

$$I = a^2 E^2 I^2 \frac{\pi^8}{l^8} \left[\frac{l}{2} - \frac{l}{4\pi} (\sin 2\pi - \sin 0) \right] + w^2 l + 2 a EI w \frac{\pi^4}{l^4} \cdot \frac{l}{\pi} (\cos \pi - \cos 0).$$

$\sin 2\pi = 0.$ $\cos \pi = -1$

$\sin 0 = 0.$ $\cos 0 = 0.$

$$I = a^2E^2I^2\frac{\pi^8}{l^8}\left(\frac{l}{2}\right) + w^2 l - 4aEI.w.\frac{\pi^3}{l^3}$$

$$\frac{\partial I}{\partial a} = 0.$$

$$2aE^2I^2\frac{\pi^8}{2l^7} - 4EIw.\frac{\pi^3}{l^3} = 0.$$

$$aE^2I^2\frac{\pi^{\cancel{8}5}}{l^4} = 4EIw.$$

$$\boxed{a = \frac{4wl^4}{EI\pi^5}.} \longrightarrow 5.$$

$$y = \frac{4wl^4}{\pi^5 EI}\sin\frac{\pi x}{l} \qquad x = l/2.$$

$$y_{max} = \frac{4wl^4}{\pi^5 EI.}$$

$$y_{max} = \frac{0.0131\, wl^4}{EI.} \longrightarrow 6.$$

(iv) Galerkin Method : $y = a\sin\left(\frac{\pi x}{l}\right).$

$$\int_0^l w_i\, R\, dx = 0.$$

$$\int_0^l \left[\left(a\sin\frac{\pi x}{l}\right)\cdot\left(aEI.\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - w.\right)\right]dx = 0.$$

$$\int_0^l \left[a^2EI.\frac{\pi^4}{l^4}\sin^2\frac{\pi x}{l} - aw\sin\frac{\pi x}{l}\right]dx = 0.$$

$$\int_0^l \left[a^2EI\frac{\pi^4}{l^4}\left(\frac{1-\cos\frac{2\pi x}{l}}{2}\right) - aw\sin\frac{\pi x}{l}\right]dx = 0.$$

$$\left[a^2 EI \frac{\pi^4}{l^4}\left(\frac{1}{2}x - \sin\frac{2\pi x}{l} \times \frac{l}{2\pi} \times \frac{1}{2}\right) + aw.\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l = 0.$$

$$a^2 EI.\frac{\pi^4}{l^3}\left(\frac{l}{2}\right) - 2aw.\left(\frac{l}{\pi}\right) = 0.$$

$$aEI.\frac{\pi^4}{l^3}\left(\frac{1}{2}\right) = 2w\left(\frac{l}{\pi}\right).$$

$$\boxed{a = \frac{4w}{EI}\left(\frac{l^4}{\pi^5}\right)} \qquad y = \frac{4w}{EI}\left(\frac{l^4}{\pi^5}\right)\sin\frac{\pi x}{l}.$$

$$y_{max} = \frac{0.01311\, wl^4}{EI} \longrightarrow (7).$$

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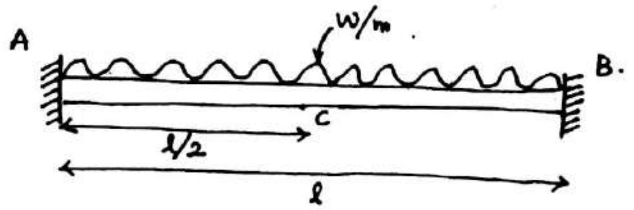
4. Find the deflection at the centre of a clamped beam subjected to UDL. Find using point collocation method. Take trail soln $y = a(x^5 - 2lx^4 + l^2x^3)$.



Sol :- Governing eqn :- $\boxed{EI\dfrac{d^4y}{dx^4} - w = 0.} \Rightarrow (1).\quad 0 \le x \le l.$

$x = 0 \qquad x = l.$

$$y = a\left(x^5 - 2lx^4 + l^2x^3\right).$$

$$\frac{dy}{dx} = a\left(5x^4 - 8lx^3 + 3l^2x^2\right).$$

$$\frac{d^2y}{dx^2} = a\left(20x^3 - 24lx^2 + 6l^2x\right).$$

$$\frac{d^3y}{dx^3} = a\left(60x^2 - 48lx + 6l^2\right).$$

$$\frac{d^4y}{dx^4} = a\,(120x - 48\ell).$$

Sub the value in eqn ①.

$$EI\,[a(120x - 48\ell)] - w = 0.$$

$$R = aEI(120x - 48\ell) - w.$$

$$R = 0.$$

$$aEI\,(120x - 48\ell) - w = 0. \qquad x = \ell/2.$$

$$aEI\left(120 \times \frac{\ell}{2} - 48\ell\right) - w = 0.$$

$$aEI\,(12\ell) = w.$$

$$\boxed{a = \frac{w}{12EI\cdot\ell}.} \longrightarrow ③.$$

Sub a in trial sol.

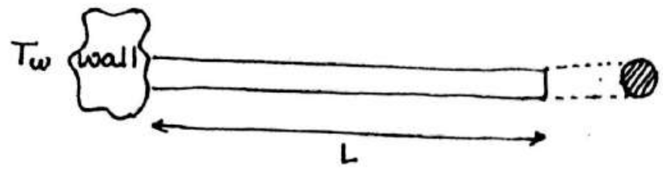
$$y = \frac{w}{12EI\ell}\left(x^5 - 2\ell x^4 + \ell^2 x^3\right). \qquad x = \ell/2.$$

$$= \frac{w}{12EI\ell}\left[\left(\frac{\ell}{2}\right)^5 - 2\ell\left(\frac{\ell}{2}\right)^4 + \ell^2\cdot\left(\frac{\ell}{2}\right)^3\right].$$

$$= \frac{w}{12EI\ell}\left[\frac{\ell^5}{32} - \frac{\ell^5}{8} + \frac{\ell^5}{8}\right].$$

$$y_{max} = \frac{w\ell^4}{384\,EI}.$$

5. Consider a 1 mm dia, 50 mm long aluminium pin-fin as shown in fig used to enhance the heat transfer from a surface wall maintained at 300°c. The governing diff eqn + the boundary conditions are given below.



$$k.\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty).$$

$T(0) - T_w = 300°c$

$\frac{dT}{dx}(L) = 0$ (insulated tip).

$K \rightarrow$ Thermal conductivity.

$P \rightarrow$ Perimeter

$A \rightarrow$ cross sectional area.

$k \rightarrow 200$ W/m°c

$h \rightarrow$ convective heat transfer coefficient

$T_w \rightarrow$ wall temp.

$T_\infty \rightarrow$ Ambient temp.

$h = 20$ W/m² °c.

$T_\infty = 30°c$

Estimate the temp. distribution in the fin using the Galerkin method.

Sol :- $d = 1\text{ mm} = 1 \times 10^{-3}$ m

$$k.\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty).$$

$l = 50\text{ mm} = 50 \times 10^{-3}$ m

$T_w = 300°c$.

B.C $\rightarrow$ (i) $T(0) = T_w = 300°c$.

$x = 0 \quad T = 300°c$

(ii) $\frac{dT}{dx}(L) = 0$.

$x = L. \quad \frac{dT}{dx} = 0$

Assume trial soln.

$$\boxed{T(x) = a_0 + a_1 x + a_2 x^2} \longrightarrow (1)$$

Apply B.C in eqn (1).

(i) $x = 0 \quad T = 300.$

$$\boxed{a_0 = 300.}$$

(ii) $x = L \quad \frac{dT}{dx} = 0.$

$$\boxed{\frac{dT}{dx} = a_1 + 2a_2 x} \longrightarrow (2).$$

$$a_1 + 2a_2(L) = 0.$$

$$\boxed{a_1 = -2a_2 L.} \longrightarrow (3).$$

Sub a_1 + a_0 values in eqn (1).

$$T(x) = 300 - (2a_2 L)x + a_2 x^2.$$

$$\boxed{T(x) = 300 + a_2(x^2 - 2Lx)} \longrightarrow (4).$$

$$K.\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty) \qquad \boxed{P = \pi D}.$$

$$200 \times \frac{d^2T}{dx^2} = \frac{\pi \times (10^{-3}) \times 20.}{\frac{\pi}{4} \times (1 \times 10^{-3})^2}(T - 30).$$

$$200 \times \frac{d^2T}{dx^2} = \frac{0.06283}{7.8539 \times 10^{-7}}(T - 30).$$

$$200.\frac{d^2T}{dx^2} = 79997.64\,(T - 30)$$

$\sim 80000.$

$$\frac{d^2T}{dx^2} = 400.(T - 30) \longrightarrow (5).$$

Sub 'T' value in eqn (5).

$$\frac{d^2T}{dx^2} = 400\left[300 + a_2(x^2 - 2Lx) - 30\right]$$

$$\frac{d^2T}{dx^2} = 400\left[270 + a_2(x^2 - 2Lx)\right] \longrightarrow (6).$$

From eq (2) $\frac{dT}{dx} = a_1 + 2a_2 x$.

$$\frac{d^2T}{dx^2} = 2a_2 \longrightarrow (7).$$

Sub $\frac{d^2T}{dx^2}$ value in eq (6).

$$2a_2 = 400\left[270 + a_2(x^2 - 2Lx)\right]$$

$$2a_2 - 400\left[270 + a_2(x^2 - 2Lx)\right] = 0. \Rightarrow \text{Governing eqn.}$$

Take $w_i = x^2 - 2Lx$.

$$\int_0^L w_i R\, dx = 0.$$

$$\int_0^L (x^2 - 2Lx)\left[2a_2 - 400\left[270 + a_2(x^2 - 2Lx)\right]\right] dx = 0.$$

$$\int_0^L (x^2 - 2Lx)(2a_2 - 10800 - 400a_2x^2 + 800a_2Lx)\, dx = 0.$$

$$\int_0^L (2a_2x^2 - 10800x^2 - 400a_2x^4 + 800a_2Lx^3 - 4a_2Lx + 216000Lx + 800a_2Lx^3 - 1600a_2L^2x^2)\, dx = 0.$$

$$\left[2a_2\frac{x^3}{3} - 10800\frac{x^3}{3} - 400a_2\frac{x^5}{5} + 800a_2L\frac{x^4}{4} - 4a_2L\frac{x^2}{2} + 216000L\frac{x^2}{2} + 800a_2L\frac{x^4}{4} - 1600a_2L^2\frac{x^3}{3}\right]_0^L = 0.$$

$$2a_2\frac{L^3}{3} - 108000\frac{L^3}{3} - 400a_2\frac{L^5}{5} + 800a_2\frac{L^4}{4} - 4a_2\frac{\ell^3}{2}$$
$$+ 216000\frac{L^3}{2} + 800a_2\frac{L^5}{4} - 1600a_2\frac{L^5}{3} = 0.$$

$$\ell^3\left[\frac{2a_2}{3} - \frac{108000}{3} - 400a_2\frac{\ell^2}{5} + 800a_2\frac{\ell^2}{4} - 2a_2 + 108000 + 200a_2\ell^2 - 1600a_2\frac{\ell^2}{3}\right] = 0.$$

$$a_2\left[0.6667 - 80\ell^2 + 200\ell^2 - 2 + 200\ell^2 - 533.33\ell^2\right] = 0.$$

$\ell = 50\times10^{-3}$ m Given

$$a_2\left[0.6667 - 0.2 + 0.5 - 2 + 0.5 - 1.333\right] = -72000.$$

$$\boxed{a_2 = 38572.81}$$

Sub a_2 value in trial sol. eq ④.

$$T(x) = 300 + 38572.81\,(x^2 - 2Lx)$$

RAYLEIGH - RITZ METHOD :

(i) integral Approach used for solving complex structural problems encountered in finite element Analysis.

(ii) Mainly used for solving solid Mechanics problem.

$$\boxed{\pi = U - H}.$$

$\pi \rightarrow$ Total Potential energy

$U \rightarrow$ Strain energy.

$H \rightarrow$ work done by ext forces.

$$\boxed{y = a_1\sin\frac{\pi x}{\ell} + a_2\sin\frac{3\pi x}{\ell} + \ldots}$$

$a_1, a_2 \rightarrow$ unknown Ritz parameters.

Accuracy of the sum depends on the Ritz parameter.

Two following conditions must be fulfilled by the Approximating function.

(i) Satisfy the geometric boundary conditions.

(ii) Function must have atleast one ritz parameter.

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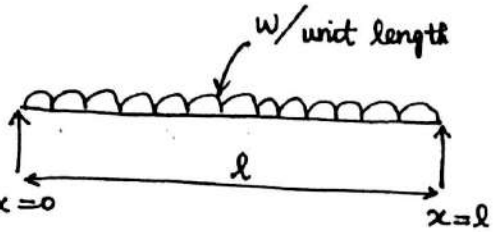
①. A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at the mid span by using Rayleigh - Ritz method + compare with exact soln.



Sol :

SSB $\longrightarrow$ Fourier Series.

$$y = \sum_{n=1,3}^{\infty} a \sin \frac{n\pi x}{l} \longrightarrow \text{Approximating function}$$

To make it simple $\rightarrow$ consider only 2 terms.

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} \longrightarrow ①.$$

$a_1, a_2 \rightarrow$ Ritz parameters.

$$\boxed{\pi = U - H} \longrightarrow ②.$$

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \longrightarrow ③.$$

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}$$

$$\frac{dy}{dx} = a_1 \cos\left(\frac{\pi x}{l}\right) \times \left(\frac{\pi}{l}\right) + a_2 \cos\left(\frac{3\pi x}{l}\right) \times \frac{3\pi}{l}$$

$$\frac{d^2y}{dx^2} = -a_1 \sin\left(\frac{\pi x}{l}\right)\left(\frac{\pi^2}{l^2}\right) - a_2 \sin\left(\frac{3\pi x}{l}\right)\left(\frac{9\pi^2}{l^2}\right) \longrightarrow (4)$$

$$U = \frac{EI}{2} \int_0^l \left[-a_1\left(\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right) - a_2\left(\frac{9\pi^2}{l^2}\right)\sin\left(\frac{3\pi x}{l}\right)\right]^2 dx$$

$$= \frac{EI}{2} \int_0^l \left[a_1 \cdot \frac{\pi^2}{l^2} \sin\left(\frac{\pi x}{l}\right) + a_2 \cdot \frac{9\pi^2}{l^2} \sin\left(\frac{3\pi x}{l}\right)\right]^2 dx$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4} \int_0^l \left[a_1 \sin\left(\frac{\pi x}{l}\right) + a_2 . 9 . \sin\left(\frac{3\pi x}{l}\right)\right]^2 dx$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4} \int_0^l \left[a_1^2 \sin^2\left(\frac{\pi x}{l}\right) + 81 . a_2^2 \sin^2\left(\frac{3\pi x}{l}\right) + 18 a_1 a_2 \sin \frac{\pi x}{l} . \sin\frac{3\pi x}{l}\right] dx$$

$$\Rightarrow \int_0^l a_1^2 \sin^2 \frac{\pi x}{l} = a_1^2 \int_0^l \left(1 - \cos \frac{2\pi x}{l}\right) \times \frac{1}{2}\, dx$$

$$= \frac{a_1^2}{2} \int_0^l \left(1 - \cos \frac{2\pi x}{l}\right) dx$$

$$= \frac{a_1^2}{2}\left([x]_0^l - \left[\sin \frac{2\pi x}{l}\right]_0^l \times \frac{l}{2\pi}\right)$$

$$= \frac{a_1^2}{2}\left(l - \frac{l}{2\pi}[\sin 2\pi - \sin 0]\right)$$

$$\int_0^l a_1^2 \sin^2 \frac{\pi x}{l} . dx = \frac{a_1^2}{2} . l \longrightarrow (6)$$

$$\Rightarrow \int_0^l 81 a_2^2 \sin^2 \frac{3\pi x}{l}\, dx = 81 a_2^2 \int_0^l \frac{1}{2}\left[1 - \cos\frac{6\pi x}{l}\cdot\frac{l}{6\pi}\right] dx$$

$$= 81 \cdot \frac{a_2^2}{2}\left([x]_0^l - \frac{l}{6\pi}\left[\overset{\sin}{\cancel{\cos}}\ \frac{6\pi x}{l}\right]_0^l \right).$$

$$= 81 \cdot \frac{a_2^2}{2}\left(l - \frac{l}{6\pi}\left[\sin 6\pi - \sin 0\right]\right).$$

$$\int_0^l 81 \cdot a_2^2 \sin^2\left(\frac{3\pi x}{l}\right) dx = 81 \cdot \frac{a_2^2}{2} \cdot l \longrightarrow (7).$$

$$\Rightarrow \int_0^l 18 a_1 a_2 \sin\frac{\pi x}{l} \sin\frac{3\pi x}{l}\, dx = 18 a_1 a_2 \int_0^l \sin\frac{\pi x}{l} \cdot \sin\frac{3\pi x}{l}\, dx.$$

$$\boxed{\therefore \sin A \cdot \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]}$$

$$= 18 a_1 a_2 \int_0^l \frac{\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}}{2}\, dx$$

$$= \frac{18 a_1 a_2}{2}\left(\left[\sin\left(\frac{2\pi x}{l}\right)\right]_0^l \times \frac{l}{2\pi} - \left[\sin\frac{4\pi x}{l}\right]_0^l \times \frac{l}{4\pi}\right).$$

$$= 9 a_1 a_2 \left[\frac{l}{2\pi}(\sin 2\pi - \sin 0) - \frac{l}{4\pi}(\sin 4\pi - \sin 0)\right].$$

$$= 0.$$

$$\int_0^l 18 a_1 a_2 \sin\frac{\pi x}{l} \cdot \sin\frac{3\pi x}{l} = 0. \longrightarrow (8).$$

$$\therefore \quad U = \frac{EI}{2} \times \frac{\pi^4}{l^4}\left(\frac{a_1^2 l}{2} + \frac{81 a_2^2}{2} l + 0\right).$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4} \cdot \frac{l}{2}\left(a_1^2 + 81 a_2^2\right).$$

$$U = \frac{EI}{4} \times \frac{\pi^4}{l^3}\left[a_1^2 + 81a_2^2\right]. \longrightarrow (9).$$

$$H = \int_0^l wy\,dx = \int_0^l w\left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right)dx.$$

$$= w\left(a_1\left[-\cos\frac{\pi x}{l}\right]_0^l \times \frac{l}{\pi} + a_2\left[-\cos\frac{3\pi x}{l}\right]_0^l\right).$$

$$= w\left[-a_1(\cos\pi - \cos 0)\frac{l}{\pi} - a_2(\cos 3\pi - \cos 0)\frac{l}{3\pi}\right].$$

$$= w.\left[\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi}\right].$$

$$= \frac{2wl}{\pi}\left[a_1 + \frac{a_2}{3}\right].$$

$$H = \frac{2wl}{\pi}\left[a_1 + \frac{a_2}{3}\right]. \longrightarrow (10).$$

Sub (9) + (10) in (2).

$$\pi = U - H.$$

$$\pi = \frac{EI}{4} \times \frac{\pi^4}{l^3}\left[a_1^2 + 81a_2^2\right] - \frac{2wl}{\pi}\left[a_1 + \frac{a_2}{3}\right]. \longrightarrow (11).$$

Find a_1 + a_2.

$$\frac{\partial\pi}{\partial a_1} = 0 \qquad \frac{\partial\pi}{\partial a_2} = 0.$$

$$\frac{\partial\pi}{\partial a_1} = \frac{EI}{4} \times \frac{\pi^4}{l^3}\, 2a_1 - \frac{2wl}{\pi} = 0.$$

$$a_1 = \frac{2wl}{\pi \times \pi^4} \times \frac{2l^3}{EI}.$$

$$\boxed{a_1 = \frac{4wl^4}{EI\,\pi^5}} \longrightarrow$$

$$\frac{\partial \pi}{\partial a_2} = 162\, a_2 \frac{EI}{4} \times \frac{\pi^4}{l^3} - \frac{2wl}{3\pi} = 0.$$

$$40.5.\, a_2\, EI.\, \frac{\pi^4}{l^3} = \frac{2wl}{3\pi}$$

$$a_2 = \frac{2wl}{3\pi} \times \frac{l^3}{40.5\, EI\, \pi^4}$$

$$\boxed{a_2 = \frac{0.01646\, wl^4}{EI\ \pi^5}}$$

Sub a_1, a_2 in trial soln.

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}.$$

$$y = \frac{4wl^4}{EI\ \pi^5} \sin \frac{\pi x}{l} + \frac{0.01646\, wl^4}{EI\ \pi^5} \sin \frac{3\pi x}{l}.$$

Sub $x = l/2$.

$$y_{max} = \frac{4wl^4}{EI\pi^5} \sin\left(\frac{\pi}{l} \times \frac{l}{2}\right) + \frac{0.01646\, wl^4}{EI\ \pi^5} \sin\left(\frac{3\pi}{l} \times \frac{l}{2}\right).$$

$$= \frac{4wl^4}{EI\pi^5} - \frac{0.01646\, wl^4}{EI\ \pi^5}.$$

$$= \frac{wl^4}{EI\pi^5}(4 - 0.01646).$$

$$y_{max} = \frac{0.0130.\, wl^4}{EI} \longrightarrow (13).$$

Exact sol:

w.k.t SSB $\rightarrow$ UDL $\rightarrow$ Max deflection.

$$y_{max} = \frac{5}{384} \frac{wl^4}{EI}.$$

$$y_{max} = 0.0130 \frac{wl^4}{EI}. \longrightarrow (14).$$

Bending Moment at mid span :-

$$M = EI \frac{d^2y}{dx^2} \longrightarrow (15).$$

From (4) sub $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = -a_1 \frac{\pi^2}{l^2} \sin \frac{\pi x}{l} - a_2 \frac{9\pi^2}{l^2} \sin \frac{3\pi x}{l}.$$

Max bending $\longrightarrow x = l/2$.

$$\frac{d^2y}{dx^2} = -a_1 \cdot \frac{\pi^2}{l^2} \sin\left(\frac{\pi}{l} \cdot \frac{l}{2}\right) - a_2 \cdot \frac{9\pi^2}{l^2} \sin\left(\frac{3\pi}{l} \cdot \frac{l}{2}\right).$$

$$= -a_1 \cdot \frac{\pi^2}{l^2} + 9 \cdot a_2 \frac{\pi^2}{l^2}.$$

$$\frac{d^2y}{dx^2} = \frac{\pi^2}{l^2}\left[-a_1 + 9a_2\right]. \longrightarrow (16).$$

Sub a_1 + a_2 values in (16).

$$\frac{d^2y}{dx^2} = \frac{\pi^2}{l^2}\left[\frac{-4wl^4}{EI\,\pi^5} + 9 \times \frac{0.01646\,wl^4}{EI \cdot \pi^5}\right].$$

$$= \frac{wl^2}{\pi^3}(-4 + 0.1481).$$

$$\boxed{\frac{d^2y}{dx^2} = -0.124 \frac{wl^2}{EI}}.$$

Sub $\frac{d^2y}{dx^2}$ in (15).

$$M_{centre} = EI \times -0.124 \frac{wl^2}{EI}$$

$$= -0.124\,wl^2. \quad [\text{-ve indicates} \downarrow \text{load}].$$

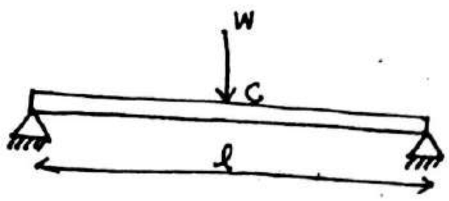
W.K.T SSB $\rightarrow$ UDL $\rightarrow$ Max bending moment.

$$M_{centre} = \frac{wl^2}{8}.$$

$$\boxed{M_{centre} = 0.125\, w l^2}$$

RESULT : Exact sol + Approx sol are almost same inorder to obtain more accurate result, more terms in Fourier series should be taken.

② A Beam AB of span 'l' SSB at ends and carrying a concentrated load W at the centre "C". Determine the deflection at the midspan by using Rayleigh-Ritz method + compare with exact solution.



SOL :-

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \cdot \sin\frac{3\pi x}{l} \longrightarrow ①$$

$$\pi = U - H \longrightarrow ②$$

$$U = \frac{EI}{2} \cdot \int_0^l \frac{d^2y}{dx^2}\, dx \longrightarrow ③$$

From eqn ⑨ Get the value of 'U'.

$$U = \frac{EI}{4} \times \frac{\pi^4}{l^3} \cdot \left[a_1^2 + 81 a_2^2\right] \longrightarrow ④$$

$$H = w\, y_{max} \longrightarrow ⑤$$

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \cdot \sin\frac{3\pi x}{l}$$

$x = l/2 \longrightarrow$ Max deflection.

$$y_{max} = a_1 \sin\left(\frac{\pi}{l} \cdot \frac{l}{2}\right) + a_2 \sin\left(\frac{3\pi}{l} \cdot \frac{l}{2}\right)$$

$$= a_1 - a_2 \longrightarrow ⑥$$

Sub (6) in (5)

$H = w(a_1 - a_2) \longrightarrow (7).$

Sub (4) + (7) in (3).

$\pi = U - H.$

$$\pi = \frac{EI}{4} \times \frac{\pi^4}{l^3}\left[a_1^2 + 81a_2^2\right] - w(a_1 - a_2) \longrightarrow (8).$$

Find a_1 + a_2 values

$$\frac{\partial \pi}{\partial a_1} = 0. \qquad \frac{\partial \pi}{\partial a_2} = 0.$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI}{4} \times \frac{\pi^4}{l^3} \times 2a_1 - w = 0.$$

$$\frac{EI}{4} \cdot \frac{\pi^4}{l^3} \times 2a_1 = w.$$

$$\boxed{a_1 = \frac{2wl^3}{EI\,\pi^4}}$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI}{4} \times \frac{\pi^4}{l^3} \times 162\,a_2 + w = 0.$$

$$a_2 = \frac{-wl^3}{EI\pi^4 \times 40.5} = \frac{-0.02469\,wl^3}{\pi^4 \cdot EI}.$$

Max deflection : $y_{max} = a_1 - a_2.$

$$y_{max} = \frac{2wl^3}{\pi^4 EI} + \frac{0.02469\,wl^3}{\pi^4\, EI}$$

$$= \frac{wl^3}{\pi^4 EI}\left[2 + 0.02469\right].$$

$$= \frac{0.02078\,wl^3}{EI}. \longrightarrow \text{Approx soln}$$

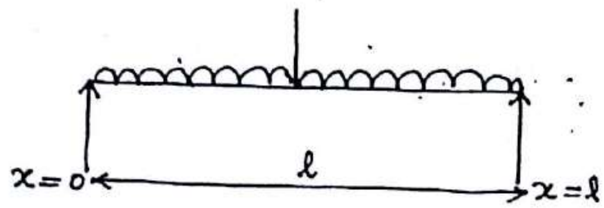
SSB $\longrightarrow$ point load at centre.

$$y_{max} = \frac{wl^3}{48EI} \longrightarrow \text{Exact soln.}$$

$\therefore$ w.k.T Approx sol + Exact sol are almost same.

For Accurate result, we can take more terms in Fourier series.

③. A SSB with UDL entire span + it is subjected to point load at the centre of span. Calculate the bending moment and deflection at mid span by using Rayleigh Ritz method + compare with exact soln.



From eqn ①.

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \cdot \sin\frac{3\pi x}{l} \longrightarrow ①.$$

$$\pi = U - H. \longrightarrow ②.$$

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

From eqn ⑨.

$$U = \frac{EI}{4} \times \frac{\pi^4}{l^3}\left[a_1^2 + 81a_2^2\right] \longrightarrow ③.$$

$$H = \int_0^l wy\,dx + wy_{max}. \longrightarrow ④.$$

From eq ⑩.

$$\int_0^l wy\,dx = \frac{2wl}{\pi}\left[a_1 + \frac{a_2}{3}\right] \longrightarrow ⑤.$$

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \cdot \sin\frac{3\pi x}{l}.$$

Max deflection, $x = l/2$.

$$y_{max} = a_1 - a_2$$

$$H = \int_0^l wy\,dx + w y_{max}$$

$$H = \frac{2wl}{\pi}\left(a_1 + \frac{a_2}{3}\right) + w(a_1 - a_2). \longrightarrow (7).$$

Sub (5) + (7) in (2).

$$\pi = \frac{EI}{4} \times \frac{\pi^4}{l^3}\left[a_1^2 + 81 a_2^2\right] - \frac{2wl}{\pi}\left(a_1 + \frac{a_2}{3}\right) - w(a_1 - a_2). \longrightarrow (8)$$

$$\frac{\partial \pi}{\partial a_1} = 0. \qquad \frac{\partial \pi}{\partial a_2} = 0.$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI}{4} \times \frac{\pi^4}{l^3} \times 2a_1 - \frac{2wl}{\pi} - w = 0.$$

$$\frac{EI}{2} \times \frac{\pi^4}{l^3} a_1 = \frac{2wl}{\pi} + w.$$

$$\boxed{a_1 = \frac{2l^3}{EI\pi^4}\left(\frac{2wl}{\pi} + w\right)} \longrightarrow (9).$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI}{4} \frac{\pi^4}{l^3}(162 a_2) - \frac{2wl}{3\pi} + w.$$

$$\frac{EI}{4} \cdot \frac{\pi^4}{l^3}(162 a_2) = \frac{2wl}{3\pi} - w.$$

$$\boxed{a_2 = \frac{2l^3}{81 EI \pi^4}\left(\frac{2wl}{3\pi} - w\right)} \longrightarrow (10).$$

Sub $a_1 + a_2$ in (6).

$$y_{max} = a_1 - a_2.$$

$$= \frac{2l^3}{EI\pi^4}\left(\frac{2wl}{\pi} + w\right) - \frac{2l^3}{81EI\pi^4}\left(\frac{2wl}{3\pi} - w\right).$$

$$= \frac{4wl^4}{EI\pi^5} + \frac{2wl^3}{EI\pi^4} - \frac{4wl^4}{243EI\pi^5} + \frac{2wl^3}{81EI\pi^4}$$

$$= \frac{4wl^4}{EI.\pi^5}\left(1 - \frac{1}{243}\right) + \frac{2wl^3}{EI\pi^4}\left(1 + \frac{1}{81}\right).$$

$$y_{max} = \frac{0.01030\, wl^4}{EI} + \frac{0.0207\, wl^3}{EI.} \longrightarrow (11).$$

SSB $\rightarrow$ UDL $\rightarrow$ $y_{max} = \frac{5}{384}\frac{wl^4}{EI.}$

SSB $\rightarrow$ Point load $\longrightarrow$ $y_{max} = \frac{wl^3}{48EI}.$

Total deflection, $y_{max} = \frac{5}{384}\cdot\frac{wl^4}{EI} + \frac{wl^3}{48EI}.$

$$= 0.0130\,\frac{wl^4}{EI} + 0.0208\,\frac{wl^3}{EI.} \longrightarrow (12).$$

Bending Moment at Mid span :

$$M = EI.\frac{d^2y}{dx^2}. \longrightarrow (13).$$

Sub $\frac{d^2y}{dx^2} = -\left[a_1\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + a_2\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right].$

Sub a_1 & a_2 values in above.

$$= -\left[\frac{2l^3}{EI\pi^4}\left(\frac{2wl}{\pi} + w\right).\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{2l^3}{81EI\pi^4}\left(\frac{2wl}{3\pi} - w\right)\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right].$$

For max deflection $\longrightarrow x = l/2.$

$$\frac{d^2y}{dx^2} = -\left[\frac{2l}{EI\pi^2}\left(\frac{2wl}{\pi} + w\right)\sin\left(\frac{\pi}{l}\cdot\frac{l}{2}\right) + \frac{2l}{9EI\pi^2}\left(\frac{2wl}{3\pi} - w\right)\sin\left(\frac{3\pi}{l}\cdot\frac{l}{2}\right)\right]$$

$\sin\frac{\pi}{2} = 1 \qquad \sin\frac{3\pi}{2} = -1.$

$$= -\left[\frac{2l}{EI\pi^2}\left(\frac{2wl}{\pi} + w\right) - \frac{2l}{9EI\pi^2}\left(\frac{2wl}{3\pi} - w\right)\right].$$

$$= -\left[\frac{4wl^2}{EI\pi^3} + \frac{2wl}{EI\pi^2} - \frac{4wl^2}{27EI\pi^3} + \frac{2wl}{9EI\pi^2}\right].$$

$$= -\left[\frac{4wl^2}{EI\pi^3}\left(1 - \frac{1}{27}\right) + \frac{2wl}{EI\pi^2}\left(\frac{1}{9} + 1\right)\right]$$

$$\frac{d^2y}{dx^2} = -\left[0.124\frac{wl^2}{EI} + 0.225.\frac{wl}{EI}\right] \longrightarrow (14).$$

Sub (14) in (13).

$$M_{centre} = -EI\left[0.124\frac{wl^2}{EI} + 0.225.\frac{wl}{EI}\right]$$

$$= -\left(0.124wl^2 + 0.225wl\right) \longrightarrow \text{Approx. } (15)$$

$SSB \rightarrow UDL \rightarrow M_{centre} = \frac{wl^2}{8}.$

$SSB \rightarrow$ point load $\rightarrow M_{centre} = \frac{wl}{4}.$

Total Bending, $M_{centre} = \frac{wl^2}{8} + \frac{wl}{4}.$

$$M_{centre} = 0.125wl^2 + 0.25wl \longrightarrow (16).$$

Approx + exact sol are almost same.

solve $\frac{d^2y}{dx^2} + 100 = 0 \qquad 0 \le x \le 10.$

$y(0) = 0 \qquad y(10) = 0.$

Using (i) point collocation method.

(ii) Sub domain method

(iii) Least square method

(iv) Galerkin's method.

Assume trial soln.

$$y = a_1 x(10-x) \longrightarrow ①.$$

$$\left.\begin{array}{ll} x=0 & y=0 \\ x=10 & y=0. \end{array}\right\} \text{trial sol satisfies B.C.}$$

(i) <u>Point Collocation method</u> :-

$$y = a_1 x(10-x).$$

$$y = a_1(10x - x^2).$$

$$\frac{dy}{dx} = a_1(10-2x). \qquad \frac{d^2y}{dx^2} = a_1(-2).$$

$$\frac{d^2y}{dx^2} = -2a_1 \longrightarrow ②.$$

Sub ② in governing eqn.

$$R = -2a_1 + 100.$$

$$\boxed{R = 0}.$$

$$-2a_1 + 100 = 0.$$

$$\boxed{a_1 = 50}.$$

$$\boxed{y = 50x(10-x)}$$

(v) Sub domain method :-

$$\int_0^{10} R\,dx = 0.$$

$$\int_0^{10} (-2a_1 + 100)\,dx = 0.$$

$$\left[-2a_1 x + 100x\right]_0^{10} = 0.$$

$$-20a_1 + 1000 = 0.$$

$$\boxed{a_1 = 50}.$$

$$y = 50x(10-x) \longrightarrow (4).$$

(vi) Least Square method :-

$$I = \int_0^{10} R^2 dx$$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} R \cdot \frac{\partial R}{\partial a_1}\,dx$$

$$R = -2a_1 + 100.$$

$$\frac{\partial R}{\partial a_1} = -2.$$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} (-2a_1 + 100) - 2 \cdot dx$$

$$\frac{\partial I}{\partial a_1} = 0.$$

$$\int_0^{10} (4a_1 - 200)\,dx = 0 \Rightarrow \left[4a_1 x - 200x\right]_0^{10} = 0.$$

$$40a_1 - 2000 = 0$$

$$\boxed{a_1 = 50}.$$

$$y = 50x(10-x) \longrightarrow (5).$$

(iv) Galerkin's Method :

$$\int_0^{10} w_i R\,dx = 0. \qquad y = w_i = a_1 x(10-x).$$

$$\int_0^{10} a_1 x(10-x)(-2a_1+100)\,dx = 0.$$

$$a_1 \int_0^{10} (10x - x^2)(-2a_1+100)\,dx = 0.$$

$$a_1 \int_0^{10} (-20a_1 x + 1000x + x^2 2.a_1 - 100x^2)\,dx = 0.$$

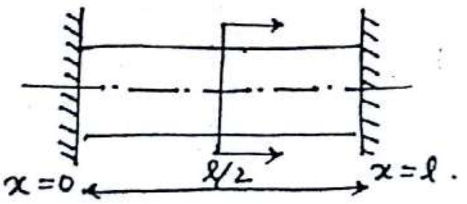
$$a_1 \left[-20.a_1 \frac{x^2}{2} + 1000\frac{x^2}{2} + a_1\frac{x^3}{3} - 100\frac{x^3}{3} \right]_0^{10} = 0.$$

$$-10 a_1 \times 10^2 + 500(10^2) + \frac{2a_1(10)^3}{3} - \frac{100(10^3)}{3} = 0.$$

$$\boxed{a_1 = 50}$$

$$\boxed{y = 50x(10-x)}$$

(5). Using Rayleigh - Ritz method, determine the expression for displacement & stress in a fixed bar subjected to axial force P.



To find : Displacement & stress variation.

Sol : Polynomial fn for 3 terms.

$$\boxed{u = a_0 + a_1 x + a_2 x^2} \longrightarrow (1).$$

$$\left.\begin{array}{ll} x=0 & u=0 \\ x=l & u=0 \end{array}\right\} \rightarrow B.C.$$

Sub B.C in ①.

$$\boxed{a_0 = 0.}$$

$$u = 0 + a_1 l + a_2 l_2.$$

$$a_1 l = -a_2 l^2$$

$$a_1 = -a_2 l.$$

sub a_0 + a_1 in eqn ①.

$$u = 0 - a_2 l x + a_2 x^2.$$

$$\boxed{u = a_2 [x^2 - l x]} \rightarrow ②.$$

$$x = l/2.$$

$$u_l = a_2 \left[\frac{l^2}{4} - l \times \frac{l}{2}\right].$$

$$= a_2 \cdot \left[\frac{l^2}{4} - \frac{l^2}{2}\right].$$

$$\boxed{u_l = -\frac{a_2 l^2}{4}} \rightarrow ⑤.$$

$$\boxed{\pi = U - H}.$$

$$U = \frac{EA}{2} \cdot \int_0^l \left(\frac{du}{dx}\right)^2 dx - P u_1 \rightarrow ④.$$

$$u = a_2 (x^2 - l x).$$

$$\frac{du}{dx} = a_2 (2x - l).$$

$$\pi = \frac{EA}{2} \cdot \int_0^l [a_2 (2x - l)]^2 dx - P\left(-\frac{a_2 l^2}{4}\right).$$

$$\pi = \frac{EA}{2} a_2^2 \int_0^l (4x^2 + l^2 - 4xl)\,dx - P\frac{a_2 l^2}{4}$$

$$= \frac{EA}{2} a_2^2 \left[\frac{4x^3}{3} + l^2 x - \frac{4x^2}{2} l\right]_0^l + a_2 \frac{l^2}{4} P.$$

$$= \frac{EA}{2} a_2^2 \left[\frac{4l^3}{3} + l^3 - 2l^3\right] + a_2 \frac{l^2}{4} P.$$

$$\pi = \frac{EA}{2} a_2^2 \times \frac{l^3}{3} + a_2 \cdot \frac{l^2}{4} P \longrightarrow (5).$$

$$\frac{\partial \pi}{\partial a_2} = 0.$$

$$\frac{EA}{2} \times 2a_2 \times \frac{l^3}{3} + \frac{l^2}{4} P = 0.$$

$$\frac{EA\, a_2 l^3}{3} = -\frac{l^2}{4} P.$$

$$a_2 = \frac{-3P}{EA\, l \cdot 4}.$$

$$\boxed{a_2 = \frac{-3P}{4EA\, l}} \longrightarrow (6).$$

Sub a_2 in trial sol.

$$u_l = \frac{3P}{4EA l} \times \frac{l^2}{4}.$$

$$\boxed{u_l = \frac{3Pl}{16EA}}.$$

Stress in the bar.

$$\sigma = E\frac{du}{dx}.$$

$$= E \cdot a_2 (2x - l).$$

$$= -E \cdot \frac{3P}{4EA \cdot l} (2x - l).$$

$$\sigma = \frac{3P}{4Al}(l-2x).$$

$x = 0.$ $\quad \sigma_0 = \sigma_{x=0.} = \frac{3P}{4Al}.$

$x = l/2.$ $\quad \sigma_l = \sigma_{x=l/2} = \frac{3P}{4A.l}\left(l - 2l/2\right).$

$x = l.$ $\quad \sigma_3 = \frac{3P}{4Al}.(l-2l).$

$$\boxed{\sigma_3 = \frac{-3P}{4A}.}$$

⑥. Solve.

$$AE \frac{d^2y}{dx^2} + q_0 = 0.$$

(i) $y(0) = 0$ (ii) $\left.\frac{dy}{dx}\right|_{x=L} = 0.$

Find the value of $f(x)$ using the weighted residual method

Sol : $AE.\frac{d^2y}{dx^2} + q_0 = 0.$

B.C $\rightarrow$ (i) $y(0) = 0.$

(ii) $\left.\frac{dy}{dx}\right|_{x=L} = 0.$

$\boxed{y(x) = a_0 + a_1 x + a_2 x^2.}$ $\longrightarrow$ Trial soln.

(i) $x = 0$ $y = 0.$

$\boxed{a_0 = 0}$

(ii) $y = a_0 + a_1 x + a_2 x^2.$

$$\frac{dy}{dx} = a_1 + 2a_2 x.$$

$x = L.$ $\quad \frac{dy}{dx} = 0.$

$$a_1 + 2a_2 L = 0 \Rightarrow a_1 = -2a_2 L.$$

Sub a_0 & a_1 in trial Soln.

$$y(x) = 0 - 2a_2Lx + a_2x^2.$$

$$y(x) = a_2[x^2 - 2Lx].$$

$$\frac{dy}{dx} = a_2[2x - 2L].$$

$$\frac{d^2y}{dx^2} = 2a_2.$$

$$\boxed{R = AE.\frac{d^2y}{dx^2} + q_0.}$$

$$R = 0.$$

$$AE.2a_2 + q_0 = 0.$$

$$\boxed{a_2 = \frac{-q_0}{2AE}.}$$

Sub a_2 in eq (2).

$$y(x) = \frac{-q_0}{2AE}.\left[x^2 - 2xL\right]..$$

$$y(x) = \frac{q_0}{2AE}.\left[2xL - x^2\right].$$

(7). $\frac{d^2y}{dx^2} + y = 4x.$ $\quad 0 \le x \le l.$

$y(0) = 0$

$y(l) = 1.$

obtain one term approx. sol by using Galerkin's method by weighted Residuals.

$$\boxed{y = a_1x(x-1) + x} \longrightarrow (1).$$

$x = 0 \quad y = 0.$

$x = 1 \quad y = 1.$

$$y = a_1 x(x-1) + x \doteq a_1(x^2 - x).$$

$$\frac{dy}{dx} = a_1(2x-1) + 1.$$

$$\boxed{\frac{d^2y}{dx^2} = 2a_1}.$$

$$2a_1 + y = 4x.$$

$$R = 2a_1 + y - 4x.$$

$$R = 2a_1 + a_1 x(x-1) + x - 4x.$$

$$\boxed{w_i = a_1(x)(x-1)}.$$

$$\int_0^l w_i R\,dx = 0.$$

$$\int_0^l a_1 x(x-1)(2a_1 + a_1(x^2 - x) - 3x)\,dx = 0.$$

$$\int_0^l (a_1x^2 - a_1x)(2a_1 + a_1x^2 - a_1x - 3x)\,dx = 0.$$

$$\int_0^l \left[2a_1^2x^2 + a_1^2x^4 - a_1^2x^2 - a_1x^3.3 - 2a_1^2x - a_1^2x^3\right]dx = 0$$

$$\boxed{a_1 = 0.833}.$$

$$y = 0.833x(x-1) + x$$

$$\boxed{y = 0.833x^2 + 0.17x}.$$

(8). The diff eqn of

$$\frac{d^2y}{dx^2} + 500x^2 = 0, \qquad 0 \le x \le l.$$

trial sol. $y = a_1(x - x^3) + a_2(x - x^5)$.

$y(0) = 0$

$y(1) = 0$.

$$\boxed{y = a_1(x - x^3) + a_2(x - x^5).} \longrightarrow \text{Trial Sol.}$$

$\longrightarrow$ (1).

B.C $\rightarrow$ $x = 0 \quad y = 0$

$x = l \quad y = 0$.

$$y = a_1(x - x^3) + a_2(x - x^5).$$

$$\frac{dy}{dx} = a_1(1 - 3x^2) + a_2(1 - 5x^4).$$

$$\frac{d^2y}{dx^2} = a_1(-6x) + a_2(-20x^3). \longrightarrow (2).$$

$$R = -6x a_1 - 20x^3 a_2 + 500x^2.$$

Limit 0 to 1 is divided into 0 to $l/2$ + $l/2$ to 1.

(1) point collocation :-

$$R = 0.$$

$$-6x a_1 - 20x^3 a_2 + 500x^2 = 0.$$

Domain 1 :- 0 to $l/2$ $\rightarrow$ Take $x = 1/3 \rightarrow 0.33$.

$$-6.\left(\frac{1}{3}\right)a_1 - 20.\left(\frac{1}{3}\right)^3 a_2 + 500\left(\frac{1}{3}\right)^2 = 0.$$

$$\boxed{-2a_1 - 0.741\, a_2 = -55.56} \longrightarrow (3).$$

Solve eq ④ + ⑤.

Domain 2 :- $l/2$ to l. $x = 2/3 \rightarrow 0.666$.

$$-6\left(\frac{2}{3}\right)a_1 - 20\left(\frac{2}{3}\right)^3 a_2 + 500\left(\frac{2}{3}\right)^2 = 0.$$

$$\boxed{-4a_1 - 5.923\,a_2 = -222.22.} \longrightarrow ④.$$

Solve eq ④ + ③.

$$\boxed{a_1 = 18.51}$$

$$\boxed{a_2 = 25.02}$$

$$y = 18.51\,(x - x^3) + 25.02\,(x - x^5) \longrightarrow ⑤.$$

②. Sub domain :- $\int_0^l R\,dx = 0$

Limit 0 to 1 $\rightarrow$ 0 to $l/2 \rightarrow l/2$ to l.

domain 1 : $\int_0^{l/2} R\,dx = 0.$

$$\int_0^{l/2} (-6a_1 x - 20a_2 x^3 + 500x^2)\,dx = 0.$$

$$-6a_1\left[\frac{x^2}{2}\right]_0^{l/2} - 20a_2\left[\frac{x^4}{4}\right]_0^{l/2} + 500\left[\frac{x^3}{3}\right]_0^{l/2} = 0.$$

$$-0.75a_1 - 0.3125\,a_2 + 20.83 = 0.$$

$$-0.75a_1 - 0.3125a_2 = -20.83. \longrightarrow ⑥.$$

domain 2 :- $\int_{l/2}^{l} (-6a_1 x - 20a_2 x^3 + 500x^2)\,dx = 0.$

$$-6a_1\left[\frac{x^2}{2}\right]_{l/2}^{l} - 20a_2\left[\frac{x^4}{4}\right]_{l/2}^{l} + 500\left[\frac{x^3}{3}\right]_{l/2}^{l} = 0.$$

$$-6a_1\left[\frac{l}{2}-\frac{(l/2)^2}{2}\right]-20a_2\left[\frac{l}{4}-\frac{(l/2)^4}{4}\right]+500\left[\frac{l}{3}-\frac{(l/2)^3}{3}\right]=0.$$

$$-2.25a_1-4.69a_2+145.83=0.$$

$$-2.25a_1-4.69a_2=-145.83. \longrightarrow (7).$$

$$\boxed{a_1=18.52}.$$

$$\boxed{a_2=22.31}.$$

$$y=18.52(x-x^3)+22.31(x-x^5).$$

(3). Least square method :

$$I=\int_0^1 R^2\,dx$$

$$\frac{\partial I}{\partial a_1}=\int_0 R\frac{\partial R}{\partial a_1}\,dx$$

domain 1 :- $\frac{\partial I}{\partial a_1}=\int_0^{1/2} R.\frac{\partial R}{\partial a_1}\,dx.$

$$R=-6a_1x-20a_2x^3+500x^2.$$

$$\frac{\partial R}{\partial a_1}=-6x.$$

$$\frac{\partial I}{\partial a_1}=\int_0^{1/2}(-6a_1x-20a_2x^3+500x^2)(-6x)\,dx=0.$$

$$\int_0^{1/2}(-6a_1x-20a_2x^3+500x^2)(-6x)\,dx=0.$$

$$\int_0^{1/2}(36a_1x^2+120a_2x^4-3000x^3)\,dx=0.$$

$$36a_1\left[\frac{x^3}{3}\right]_0^{1/2}+120.a_2\left[\frac{x^5}{5}\right]_0^{1/2}-3000\left[\frac{x^4}{4}\right]_0^{1/2}=0.$$

$$1.5a_1 + 0.75a_2 - 46.88 = 0.$$

$$1.5a_1 + 0.75a_2 = 46.88 \longrightarrow (8).$$

domain 2 :- $\dfrac{\partial I}{\partial a_2} = \displaystyle\int_{1/2}^{1} R \cdot \frac{\partial R}{\partial a_2}\, dx$

$$R = -6a_1 x - 20a_2 x^3 + 500x^2$$

$$\frac{\partial R}{\partial a_2} = -20x^3.$$

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} (-6a_1 x - 20a_2 x^3 + 500x^2)(-20x^3)\, dx$$

$$\frac{\partial I}{\partial a_2} = 0.$$

$$\int_{1/2}^{1} (-6a_1 x - 200a_2 x^3 + 500x^2)(-20x^3)\, dx = 0.$$

$$\int_{1/2}^{1/2} (120x^4 + 400a_2 x^6 - 10000x^5)\, dx = 0.$$

$$120a_1 \left[\frac{x^5}{5}\right]_{1/2}^{1} + 400a_2 \left[\frac{x^7}{7}\right]_{1/2}^{1} - 10000\left[\frac{x^6}{6}\right]_{1/2}^{1} = 0.$$

$$a_1 \cdot \frac{120}{5}\left[\left(\frac{1}{2}\right)^5 - 1^5\right] + \frac{400}{7} a_2 \left[\left(\frac{1}{2}\right)^7 - 1^7\right] - \frac{10000}{6}\left[\left(\frac{1}{2}\right)^6 - (1)^6\right] = 0.$$

$$23.25a_1 + 56.7a_2 = 1640.62 \longrightarrow (9).$$

Solve (8) + (9).

$$\boxed{a_1 = 21.11} \qquad \boxed{a_2 = 20.28}.$$

$$y = 21.11(x - x^3) + 20.28(x - x^5).$$

4. Galerkin's Method :

$$\int_0^1 w_i R\,dx = 0.$$

domain 1 :- $$\int_0^{1/2} w_i R\,dx = 0.$$

$y = w_i = x - x^3.$ $\quad R = -6a_1 x - 20a_2 x^3 + 500\,x^2.$

$$\int_0^{1/2} (x - x^3)(-6a_1 x - 20a_2 x^3 + 500\,x^2)\,dx = 0.$$

$$\int_0^{1/2} (-6a_1 x^2 - 20a_2 x^4 + 500x^3 + 6a_1 x^4 + 20a_2 x^6 - 500x^5)\,dx = 0.$$

$$-6a_1\left[\frac{x^3}{3}\right]_0^{1/2} - 20a_2\left[\frac{x^5}{5}\right]_0^{1/2} + 500\left[\frac{x^4}{4}\right]_0^{1/2} + 6a_1\left[\frac{x^5}{5}\right]_0^{1/2}$$

$$+ 20a_2\cdot\left[\frac{x^7}{7}\right]_0^{1/2} - 500\left[\frac{x^6}{6}\right]_0^{1/2} = 0.$$

$$-0.25a_1 - 0.125a_2 + 7.81 + 0.0375a_1 + 0.022a_2 - 1.303 = 0.$$

$$-0.2125a_1 - 0.103\,a_2 + 6.51 = 0.$$

$$-0.2125a_1 - 0.103\,a_2 = -6.51 \longrightarrow (10).$$

domain 2 :- $$\int_{1/2}^1 w_i R\,dx = 0.$$

$y = w_i = (x - x^5).$

$$\int_{1/2}^1 (x - x^5)(-6a_1 x - 20a_2 x^3 + 500x^2)\,dx = 0.$$

$$\int_{1/2}^1 (-6a_1 x^2 - 20a_2 x^4 + 500x^3 + 6a_1 x^6 + 20a_2 x^8 - 500x^7)\,dx = 0.$$

$$-6a_1\left[\frac{x^3}{3}\right]_{1/2}^{1} - 20a_2\left[\frac{x^5}{5}\right]_{1/2}^{1} + 500\left[\frac{x^4}{4}\right]_{1/2}^{1} + 6a_1\left[\frac{x^7}{7}\right]_{1/2}^{1}$$

$$+ 20a_2\left[\frac{x^9}{9}\right]_{1/2}^{1} - 500\left[\frac{x^8}{8}\right]_{1/2}^{1} = 0.$$

$$-\frac{6a_1}{3}\left[1^3-\left(\frac{1}{2}\right)^3\right] - \frac{20a_2}{5}\cdot\left[1^5-\left(\frac{1}{2}\right)^5\right] + \frac{500}{4}\left[1^4-\left(\frac{1}{2}\right)^4\right]$$

$$+ \frac{6a_1}{7}\left[1^7-\left(\frac{1}{2}\right)^7\right] + \frac{20a_2}{9}\left[1^9-\left(\frac{1}{2}\right)^9\right] - \frac{500}{8}\left[1^8-\left(\frac{1}{2}\right)^8\right] = 0.$$

$$-1.75a_1 - 3.875a_2 + 117.19 + 0.85a_1 + 2.22a_2 - 62.25 = 0.$$

$$-0.9a_1 - 1.655a_2 = -54.94 \longrightarrow (11).$$

solve (10) + (11).

$$\boxed{a_1 = 19.75} \qquad \boxed{a_2 = 22.45}.$$

$$y = 19.75(x - x^3) + 22.45(x - x^5).$$